



Commutativity of Toeplitz and Hankel Operators on Beurling Subspaces of the Hardy Space

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ABSTRACT

We study the commutativity of Toeplitz and Hankel operators on Beurling subspaces of the Hardy space, allowing bounded symbols with polynomial coanalytic parts. Suppose that the Toeplitz symbol has a nonzero coanalytic part of degree p and that the Hankel symbol has coanalytic degree m . We prove that the commutator vanishes on the monomial Beurling subspace $z^k H^2$ if and only if $k \geq p + m$, and show that this threshold is sharp. When the Toeplitz symbol is analytic and nonconstant, the corresponding threshold is m . We introduce the Toeplitz–Hankel commutativity depth as the first level of the Beurling chain on which the restricted commutator vanishes, assigning infinite depth when no such level exists. We also determine the rank of the restricted commutator as a function of k and establish an infinite-depth result for nonconstant analytic Toeplitz symbols when the Hankel symbol has infinitely many nonzero negative Fourier coefficients. Finally, under the hypothesis that the kernel of the Hankel operator is generated by a finite Blaschke product, we obtain a sharp vanishing threshold on the corresponding Beurling subspaces.

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1. INTRODUCTION

Toeplitz and Hankel operators are among the fundamental classes of operators in operator theory and function theory on the Hardy space H^2 . They arise naturally from multiplication operators through projection onto the analytic and anti-analytic components of $L^2(\mathbb{T})$ and have been extensively studied in connection with complex analysis, function theory, and operator algebras; see, for example, [1], [2]. Their algebraic properties, including products, commutators, and commutativity, have attracted considerable attention. In particular, commutativity problems for Toeplitz operators have a long history, beginning with the classical work of Brown and Halmos [3]. Further developments considered various symbol classes and analytic function spaces; see, for example, [4]–[6]. Hankel operators exhibit a different algebraic structure, since the classical Hankel operator maps H^2 into its orthogonal complement. Nevertheless, through the standard unitary identification of this orthogonal complement with H^2 , Hankel operators may be realized as operators on H^2 , providing a natural framework in which such algebraic relations can be studied. Commutativity and related algebraic problems for Hankel operators have also been investigated in [7], [8].

Mixed commutativity relations between Toeplitz and Hankel operators have also been widely studied. In particular, Martínez-Avendaño [9] characterized the pairs of Toeplitz and Hankel operators that commute on the Hardy space. Related investigations have considered compact mixed commutators [10] and finite-rank mixed commutators [11], as well as extensions of Toeplitz–Hankel commutativity to multivariable Hardy spaces [12], [13]. These results concern commutativity or related properties on the whole underlying space. A different question arises when the operators fail to commute on H^2 : whether their commutators may vanish after restriction to a proper invariant subspace, and, if so, at which level of a nested family of invariant subspaces this phenomenon first occurs.

A natural setting for this question is provided by Beurling subspaces of the Hardy space. These are closed subspaces of H^2 of the form θH^2 , where θ is an inner function. By Beurling's theorem [14], these are precisely the nontrivial closed invariant subspaces of the unilateral shift on H^2 . Restricted Toeplitz and Hankel operators associated with Beurling subspaces and model spaces have also been considered recently in [15]. Among these, the monomial Beurling subspaces

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$$z^k H^2, \quad k \in \mathbb{N}_0,$$

form the nested chain

$$H^2 \supset zH^2 \supset z^2H^2 \supset \dots$$

This structure makes it possible to study whether commutativity, although absent on the whole Hardy space, appears sufficiently far along the Beurling chain. Motivated by this viewpoint, we consider Toeplitz and Hankel operators with symbols having polynomial coanalytic parts and determine exactly the members $z^k H^2$ on which they commute. Here commutativity on $z^k H^2$ means the vanishing of the restricted commutator, not the commutation of the separately compressed operators.

Our main results determine the vanishing threshold of the restricted Toeplitz–Hankel commutator on monomial Beurling subspaces for bounded symbols with polynomial coanalytic parts. Suppose that the Toeplitz symbol φ has a nonzero coanalytic part of degree p and that the Hankel symbol ψ has coanalytic degree m ; $p, m \geq 1$. We prove that T_φ and H_ψ commute on $z^k H^2$ if and only if $k \geq p + m$. Moreover, this threshold is sharp, since the commutator does not vanish on $z^{p+m-1} H^2$. When φ is analytic and nonconstant, the corresponding threshold is m . To describe these results in a unified form, we introduce the Toeplitz–Hankel commutativity depth $d_{TH}(\varphi, \psi)$, defined as the first level of the Beurling chain on which the restricted commutator $[T_\varphi, H_\psi]$ vanishes, with infinite depth when no such level exists. For nonconstant Toeplitz symbols φ and nonzero Hankel operators H_ψ , both with symbols having polynomial coanalytic parts, the commutativity depth is given by

$$d_{TH}(\varphi, \psi) = d_-(\varphi) + d_-(\psi).$$

Here, $d_-(\varphi)$ and $d_-(\psi)$ denote the degrees of the coanalytic parts of the respective symbols. We also determine the rank of the restricted commutator, establish an infinite-depth result for nonconstant analytic Toeplitz symbols, and extend the vanishing-threshold result to finite Blaschke-product subspaces under a Hankel-kernel hypothesis.

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation and preliminary results for Toeplitz and Hankel operators on the Hardy space. In Section 3, we derive the action of the Toeplitz–Hankel commutator $[T_\varphi, H_\psi]$ on the standard basis and establish the sharp vanishing threshold. Section 4 determines the rank of the restricted commutator along the monomial Beurling chain, including the case of analytic Toeplitz symbols. In Section 5, we introduce the Toeplitz–Hankel commutativity depth extends the analysis to finite Blaschke-product subspaces. Section 6 provides examples illustrating the main results, and Section 7 contains the concluding remarks.

2. PRELIMINARIES

Let $\mathbb{D}$ be the open unit disc and $\mathbb{T} = \partial\mathbb{D}$. The Hardy space H^2 consists of all analytic functions

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

such that

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

The monomials $\{z^n: n \geq 0\}$ form the standard orthonormal basis of H^2 . For $k \in \mathbb{N}_0$, we consider the Beurling subspace

$$z^k H^2 = \overline{\text{span}}\{z^k, z^{k+1}, z^{k+2}, \dots\}.$$

These subspaces form the decreasing chain

$$H^2 \supset zH^2 \supset z^2H^2 \supset \dots$$

We denote by $L^\infty(\mathbb{T})$ the space of all essentially bounded measurable functions on $\mathbb{T}$, equipped with the essential supremum norm. We also denote by H^∞ the space of all bounded analytic functions on $\mathbb{D}$ with the supremum norm.

Let $P: L^2(\mathbb{T}) \rightarrow H^2$ denote the orthogonal projection. For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ is defined by

$$T_\varphi f = P(\varphi f), \quad f \in H^2.$$

For $\varphi \in L^\infty(\mathbb{T})$, we denote by $\hat{\varphi}(n)$ its n -th Fourier coefficient. The Fourier expansion of φ is given by

$$\varphi(z) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) z^n.$$

When the coanalytic part of φ is a nonzero polynomial of degree $p \geq 1$, we write

$$\varphi(z) = f(z) + \sum_{r=1}^p a_{-r} z^{-r}$$

where $f \in H^\infty$ and $a_{-p} \neq 0$. We denote the exact degree of the coanalytic part of φ by $d_-(\varphi) = p$, if φ is analytic, we set $d_-(\varphi) = 0$.

Writing $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we have, for every $n \in \mathbb{N}_0$,

$$T_\varphi z^n = \sum_{j=-\min\{p,n\}}^{\infty} a_j z^{n+j},$$

Here a_{-r} for $1 \leq r \leq p$ are the coefficients in the preceding representation of φ . In particular, if $n \geq p$, then

$$T_\varphi z^n = \sum_{j=-p}^{\infty} a_j z^{n+j}. \quad (1)$$

We realize Hankel operators as operators on H^2 . Let

$$L_-^2 = L^2(\mathbb{T}) \ominus H^2,$$

and let P_- be the orthogonal projection from $L^2(\mathbb{T})$ onto L_-^2 . Define

$$U: L_-^2 \rightarrow H^2$$

by

$$U(z^{-j}) = z^{j-1}, \quad j \geq 1.$$

For $\psi \in L^\infty(\mathbb{T})$, its Fourier expansion is given by

$$\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n.$$

When the coanalytic part of ψ is a nonzero polynomial of degree $m \geq 1$, we write

$$\psi(z) = g(z) + \sum_{j=1}^m b_{-j} z^{-j}$$

where $g \in H^\infty$ and $b_{-m} \neq 0$.

The Hankel operator H_ψ is given by

$$H_\psi = UP_-M_\psi|_{H^2}$$

where M_ψ denotes multiplication by ψ .

Let

$$\psi(z) = \sum_{j=-m}^s b_j z^j$$

be a trigonometric polynomial. When its anti-analytic part is nonzero, we assume that $m \geq 1$ and $b_{-m} \neq 0$, so that m is its exact coanalytic degree, and write

$$\psi_-(z) = \sum_{j=1}^m b_{-j} z^{-j}. \quad (2)$$

For $k \in \mathbb{N}_0$, we say that T_φ and H_ψ commute on $z^k H^2$ if

$$T_\varphi H_\psi f = H_\psi T_\varphi f \quad \text{for every } f \in z^k H^2.$$

Equivalently,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0,$$

where $[T_\varphi, H_\psi] = T_\varphi H_\psi - H_\psi T_\varphi$ and $|_{z^k H^2}$ denotes the restriction of the Toeplitz-Hankel commutator to $z^k H^2$. Throughout the paper, these two formulations will be used interchangeably. The definition concerns the restriction of the commutator to $z^k H^2$ rather than the commutator of the operators obtained by compressing T_φ and H_ψ separately to $z^k H^2$. Indeed, if P_k denotes the orthogonal projection onto $z^k H^2$ and the coanalytic part of ψ has degree m , then $P_k H_\psi|_{z^k H^2} = 0$ whenever $2k \geq m - 1$. Consequently, the compressed commutator vanishes at those levels, whereas the restricted commutator has the sharp threshold $p + m$ when the coanalytic part of φ has positive degree p . The two notions therefore can have different vanishing thresholds in general.

Our Hankel convention differs slightly from that of Martínez-Avendaño [9]. In [9], the Hankel operator with symbol f has matrix entries $\hat{f}(-(i+j))$, whereas our realization

$$H_\psi = UP_-M_\psi|_{H^2}$$

has entries $\hat{\psi}(-(i+j+1))$, for $i, j \geq 0$. Hence $H_\psi = H_{z\psi}^{MA}$, where $H_{z\psi}^{MA}$ denotes the Hankel operator with symbol f under the convention of [9]. Thus the two conventions describe the same class of Hankel operators. In particular, the case $k = 0$ of our commutativity results is consistent with the classification in [9]: within our finite-coanalytic-degree symbol class, a nonzero Hankel operator commutes with T_φ on all of H^2 only when φ is constant.

The following elementary formula will be used repeatedly.

Lemma 1. Let $\psi \in L^\infty(\mathbb{T})$ have the coanalytic part given by (2). Then, for $n \geq 0$,

$$H_\psi z^n = \begin{cases} \sum_{j=n+1}^m b_{-j} z^{j-n-1}, & 0 \leq n < m, \\ 0, & n \geq m. \end{cases}$$

In particular, $H_\psi z^{m-1} = b_{-m}$ and $z^m H^2 \subseteq \ker H_\psi$.

Proof. Suppose first that $0 \leq n < m$. Since only the negative Fourier coefficients of ψ contribute to $P_-(\psi z^n)$,

$$P_-(\psi z^n) = \sum_{j=n+1}^m b_{-j} z^{n-j}, \quad 0 \leq n < m.$$

Applying U yields

$$H_\psi z^n = \sum_{j=n+1}^m b_{-j} z^{j-n-1}.$$

Now suppose that $n \geq m$. The corresponding sum is empty, and hence $H_\psi z^n = 0$. By linearity, density, and boundedness of H_ψ , it follows that $z^m H^2 \subseteq \ker H_\psi$.

Remark 1. The operator H_ψ depends only on the anti-analytic part of its symbol; that is, $H_\psi = H_{\psi_-}$. If the anti-analytic part of ψ vanishes, then $H_\psi = 0$. In the results requiring a polynomial anti-analytic part, we assume that ψ_- is given by (2), with $m \geq 1$ and $b_{-m} \neq 0$.

3. THE MIXED COMMUTATOR

For $\varphi, \psi \in L^\infty(\mathbb{T})$ consider the Toeplitz-Hankel commutator

$$[T_\varphi, H_\psi] = T_\varphi H_\psi - H_\psi T_\varphi.$$

The following formula describes its action sufficiently far along the standard basis.

Proposition 1. For $\varphi, \psi \in L^\infty(\mathbb{T})$, where

$$\varphi(z) = f(z) + \sum_{r=1}^p a_{-r} z^{-r}, \quad f \in H^\infty, \quad a_{-p} \neq 0.$$

Suppose also that the coanalytic part of ψ is given by (2), with $p, m \geq 1$ and $b_{-m} \neq 0$. Then, for every $n \geq \max\{p, m\}$,

$$[T_\varphi, H_\psi] z^n = - \sum_{r=1}^p a_{-r} H_\psi z^{n-r}. \quad (3)$$

Proof. Since $n \geq m$, Lemma 1 gives $H_\psi z^n = 0$. Hence

$$[T_\varphi, H_\psi] z^n = -H_\psi T_\varphi z^n.$$

Since $n \geq p$, we have

$$T_\varphi z^n = z^n f(z) + \sum_{r=1}^p a_{-r} z^{n-r}.$$

Since $n \geq m$ and $f \in H^\infty$, we have $z^n f \in z^m H^2$. Hence by Lemma 1,

$$H_\psi(z^n f) = 0.$$

Consequently,

$$H_\psi T_\varphi z^n = \sum_{r=1}^p a_{-r} H_\psi z^{n-r},$$

Thus,

$$[T_\varphi, H_\psi] z^n = -H_\psi T_\varphi z^n = -\sum_{r=1}^p a_{-r} H_\psi z^{n-r},$$

which completes the proof.

Corollary 1. Under the hypotheses of Proposition 1, for every $n \geq p + m$,

$$[T_\varphi, H_\psi] z^n = 0.$$

Consequently, $[T_\varphi, H_\psi]|_{z^{p+m}H^2} = 0$.

Proof. If $n \geq p + m$ and $1 \leq r \leq p$, then

$$n - r \geq m.$$

Thus $H_\psi z^{n-r} = 0$ by Lemma 1, and equation (3) gives the result.

Proposition 2. For $\varphi, \psi \in L^\infty(\mathbb{T})$, where

$\varphi(z) = f(z) + \sum_{r=1}^p a_{-r} z^{-r}$, $f \in H^\infty$, $a_{-p} \neq 0$, and the coanalytic part of ψ is given by Equation (2), with $p, m \geq 1$ and $b_{-m} \neq 0$. Then

$$[T_\varphi, H_\psi] z^{p+m-1} = -a_{-p} b_{-m} \neq 0. \quad (4)$$

Proof. Since

$$p + m - 1 \geq \max\{p, m\},$$

Proposition 1 gives

$$[T_\varphi, H_\psi] z^{p+m-1} = -\sum_{r=1}^p a_{-r} H_\psi z^{p+m-1-r}.$$

For $1 \leq r < p$,

$$p + m - 1 - r \geq m,$$

and hence the corresponding terms vanish. Therefore

$$[T_\varphi, H_\psi] z^{p+m-1} = -a_{-p} H_\psi z^{m-1}.$$

By Lemma 1, $H_\psi z^{m-1} = b_{-m}$, 1,

which proves equation (4). $\square$

Combining the preceding corollary and Proposition 2, we see that $p + m - 1$ is the largest integer n for which

$$[T_\varphi, H_\psi] z^n \neq 0.$$

Thus the threshold $p + m$ is sharp.

4. COMMUTATIVITY ON BEURLING SUBSPACES

We now determine exactly those members of the Beurling chain on which the Toeplitz-Hankel commutator vanishes.

Theorem 1. For $\varphi, \psi \in L^\infty(\mathbb{T})$, where

$\varphi(z) = f(z) + \sum_{r=1}^p a_{-r} z^{-r}$, $f \in H^\infty$, $a_{-p} \neq 0$.

Suppose also that the coanalytic part of ψ is given by (2), with $p, m \geq 1$ and $b_{-m} \neq 0$. Then, for every $k \in \mathbb{N}_0$,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0 \iff k \geq p + m.$$

Proof. If $k \geq p + m$, then by Corollary 1 we have

$$[T_\varphi, H_\psi] z^n = 0, \quad n \geq k,$$

and therefore

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0.$$

Conversely, suppose that $k < p + m$. Then

$$p + m - 1 \geq k,$$

so $z^{p+m-1} \in z^k H^2$. Proposition 2 gives

$$[T_\varphi, H_\psi] z^{p+m-1} = -a_{-p} b_{-m} \neq 0.$$

Thus the commutator does not vanish identically on $z^k H^2$, which completes the proof.

Corollary 2. Under the hypotheses of Theorem 1, the restriction

$$[T_\varphi, H_\psi] z^{p+m-1} H^2$$

has rank exactly one.

Proof. By Corollary 1 the commutator vanishes on $z^{p+m} H^2$. Hence, its restriction to $z^{p+m-1} H^2$ has range spanned by the image of z^{p+m-1} . Proposition 2 gives

$$[T_\varphi, H_\psi] z^{p+m-1} = -a_{-p} b_{-m} \neq 0. \quad \square$$

The preceding corollary shows that the restricted commutator has rank one at the critical level $k = p + m - 1$. We now extend this observation by determining its rank for every $k \geq m$, thereby describing how the rank changes along the chain of monomial Beurling subspaces."

Theorem 2. Under the hypotheses of Theorem 1, for every integer $k \geq m$,

$$\text{rank}([T_\varphi, H_\psi]|_{z^k H^2}) = \min\{m, \max\{p + m - k, 0\}\}.$$

Proof. Since $k \geq m$, the operator H_ψ vanishes on $z^k H^2$. Thus, for every $n \geq k$,

$$[T_\varphi, H_\psi] z^n = -\sum_{r=1}^p a_{-r} H_\psi z^{n-r}.$$

If $k \geq p + m$, Corollary 1 shows that the commutator vanishes on $z^k H^2$, so the stated rank is zero.

Now suppose that $m \leq k < p + m$, and put $q = \min\{m, p + m - k\}$. For every $n \geq k$, each nonzero monomial in $H_\psi z^{n-r}$ has degree at most $m - 1 - (n - r) \leq p + m - 1 - k$ and at most $m - 1$. Hence

$$\text{ran}([T_\varphi, H_\psi]|_{z^k H^2}) \subseteq \text{span}\{1, z, \dots, z^{q-1}\},$$

which gives

$$\text{rank}([T_\varphi, H_\psi]|_{z^k H^2}) \leq q.$$

For the reverse inequality, consider the q images

$$[T_\varphi, H_\psi] z^{p+m-1-j}, \quad j = 0, 1, \dots, q-1.$$

All these input monomials belong to $z^k H^2$. Let A_{ij} be the coefficient of z^i in $[T_\varphi, H_\psi] z^{p+m-1-j}$, where $0 \leq i, j < q$. By Lemma 1,

$$A_{ij} = - \sum_{r=1}^p a_{-r} b_{-(p+m+i-j-r)},$$

where $b_{-\ell} = 0$ for $\ell \notin \{1, \dots, m\}$. Substituting $t = p - r$ yields

$$A_{ij} = - \sum_{t=0}^{p-1} a_{-(p-t)} b_{-(m+i-j+t)}.$$

If $i > j$, then $m + i - j + t > m$ for every $t \geq 0$, so $A_{ij} = 0$. If $i = j$, only $t = 0$ contributes, and therefore

$$A_{ii} = -a_{-p} b_{-m} \neq 0.$$

Thus A is upper triangular with nonzero diagonal, and

$$\det A = (-a_{-p} b_{-m})^q \neq 0.$$

The q selected images are linearly independent. Consequently,

$$\text{rank}([T_\varphi, H_\psi]|_{z^k H^2}) \geq q.$$

Combining the two bounds gives

$$\text{rank}([T_\varphi, H_\psi]|_{z^k H^2}) = q = \min\{m, p + m - k\}.$$

Together with the case $k \geq p + m$, this proves the theorem. $\square$

Corollary 3. Under the hypotheses of Theorem 1, $z^{p+m} H^2$ is the first member of the monomial Beurling chain $\{z^k H^2 : k \in \mathbb{N}_0\}$ on which T_φ and H_ψ commute.

The analytic case has a different threshold.

Theorem 3. Let $\varphi \in H^\infty$ be nonconstant, let $\psi \in L^\infty(\mathbb{T})$ have the analytic part given by (2) with $m \geq 1$ and $b_{-m} \neq 0$. Then, for every $k \in \mathbb{N}_0$,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0 \iff k \geq m.$$

Equivalently,

$$T_\varphi H_\psi = H_\psi T_\varphi \text{ on } z^k H^2 \iff k \geq m.$$

Proof. Suppose first that $k \geq m$. By Lemma 1, $H_\psi f = 0$ for every $f \in z^m H^2$. Since $\varphi \in H^\infty$, multiplication by φ maps $z^m H^2$ into itself. Hence, for every $f \in z^k H^2$,

$$H_\psi f = 0, \quad H_\psi T_\varphi f = H_\psi (\varphi f) = 0.$$

Therefore,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0.$$

To prove minimality, Lemma 1 gives $H_\psi z^{m-1} = b_{-m}$. Writing $\varphi(z) = \sum_{r=0}^\infty a_r z^r$, we obtain

$$T_\varphi H_\psi z^{m-1} = b_{-m} \varphi.$$

By Lemma 1,

$$H_\psi T_\varphi z^{m-1} = a_0 b_{-m},$$

Consequently,

$$[T_\varphi, H_\psi] z^{m-1} = b_{-m}(\varphi - \varphi(0)) \neq 0,$$

because φ is nonconstant and $b_{-m} \neq 0$.

For every $k < m$, we have $z^{m-1} \in z^k H^2$. Thus the commutator does not vanish identically on $z^k H^2$, proving the assertion.

The remaining degenerate cases are immediate. If $\varphi \equiv c$, then $T_\varphi = cI$, and hence

$$[T_\varphi, H_\psi] = 0.$$

Similarly, if $\psi_- = 0$, then $H_\psi = 0$, so the commutator vanishes on H^2 .

5. TOEPLITZ–HANKEL COMMUTATIVITY DEPTH

We introduce a quantity that records the first member of the Beurling chain on which commutativity occurs.

Definition 1. For $\varphi, \psi \in L^\infty(\mathbb{T})$, we define the Toeplitz–Hankel commutativity depth by

$$d_{TH}(\varphi, \psi) = \inf\{k \in \mathbb{N}_0 : [T_\varphi, H_\psi]|_{z^k H^2} = 0\},$$

with the convention that $\inf \emptyset = \infty$.

We now use the preceding results to determine the commutativity depth for Toeplitz and Hankel symbols with polynomial coanalytic parts.

Theorem 4. Let $\varphi, \psi \in L^\infty(\mathbb{T})$, and suppose that their coanalytic parts are polynomials. If $H_\psi = 0$, then $d_{TH}(\varphi, \psi) = 0$.

Suppose $H_\psi \neq 0$, and let $m = d_-(\psi) \geq 1$. Then

$$d_{TH}(\varphi, \psi) = \begin{cases} 0, & \varphi \text{ is constant,} \\ m, & \varphi \text{ is analytic and nonconstant,} \\ p + m, & d_-(\varphi) = p \geq 1. \end{cases}$$

Proof. If $H_\psi = 0$, the assertion is immediate. If φ is constant, then T_φ is a scalar multiple of the identity, so again the commutator vanishes on H^2 .

If φ is analytic and nonconstant, Theorem 3 gives

$$d_{TH}(\varphi, \psi) = m.$$

Finally, if $d_-(\varphi) = p \geq 1$, then by Theorem 1

$$d_{TH}(\varphi, \psi) = p + m.$$

For nonconstant Toeplitz symbols Theorem 4 takes a particularly simple form.

Corollary 4. Let $\varphi, \psi \in L^\infty(\mathbb{T})$ have polynomial coanalytic parts. Suppose that φ is nonconstant and $H_\psi \neq 0$. Then

$$d_{TH}(\varphi, \psi) = d_-(\varphi) + d_-(\psi). \quad (5)$$

Thus the commutativity depth depends only on the coanalytic degrees of the two symbols and is independent of their analytic parts and of the remaining Fourier coefficients.

Theorem 5. (Infinite depth for nonconstant analytic symbols).

Let $\varphi \in H^\infty$ be nonconstant and let $\psi \in L^\infty(\mathbb{T})$. Then

$$d_{TH}(\varphi, \psi) = d_-(\psi),$$

where $d_-(\psi) = \infty$ if ψ has infinitely many nonzero negative Fourier coefficients.

Proof. If $d_-(\psi)$ is finite, the conclusion follows from Theorem 3 (and is immediate when $H_\psi = 0$). Suppose $d_-(\psi) = \infty$ and, contrary to the claim, that for some $k \geq 0$,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0.$$

Put $h_n = H_\psi z^n$. Our Hankel convention gives

$h_{n+1} = S^* h_n$, where $S = T_z$. For every $n \geq k$, commutator vanishing yields

$$\varphi h_n = H_\psi(\varphi z^n).$$

Applying S^* to both sides and using $H_\psi S = S^* H_\psi$, we obtain

$$S^*(\varphi h_n) = H_\psi(\varphi z^{n+1}) = \varphi h_{n+1}.$$

Since φ is analytic, the backward-shift product identity gives

$$S^*(\varphi h_n) = \varphi S^* h_n + (S^* \varphi) h_n(0) = \varphi h_{n+1} + (S^* \varphi) h_n(0).$$

Consequently, $(S^* \varphi) h_n(0) = 0$ for every $n \geq k$. As φ is nonconstant, $S^* \varphi \neq 0$, and therefore

$$h_n(0) = b_{-(n+1)} = 0, \quad n \geq k.$$

Thus all negative Fourier coefficients of ψ beyond degree k vanish, contradicting $d_-(\psi) = \infty$. Hence the commutator vanishes on no monomial tail, so $d_{TH}(\varphi, \psi) = \infty$. This completes the proof. $\square$

The following result extends the vanishing-depth calculation to a Beurling subspace determined by the kernel of a Hankel operator.

Theorem 6. Let θ be a nonconstant finite Blaschke product and let $\psi \in L^\infty(T)$ satisfy $\ker H_\psi = \theta H^2$. Suppose that

$$\varphi(z) = f(z) + \sum_{r=1}^p a_{-r} z^{-r}, \quad f \in H^\infty, p \geq 1, \quad a_{-p} \neq 0.$$

Then, for every $k \in \mathbb{N}_0$,

$$[T_\varphi, H_\psi]|_{z^k \theta H^2} = 0 \quad \Leftrightarrow \quad k \geq p.$$

Proof. Fix $k \geq p$ and write $h = z^k \theta g$ with $g \in H^2$. As $h \in \ker H_\psi$,

$$[T_\varphi, H_\psi]h = -H_\psi T_\varphi h.$$

Since $k - r \geq 0$ for $1 \leq r \leq p$,

$$T_\varphi h = f h + \sum_{r=1}^p a_{-r} z^{k-r} \in \theta H^2.$$

Thus $[T_\varphi, H_\psi]h = 0$.

For sharpness, choose $h = z^{p-1} \theta$. The analytic part of φ and the terms with $r < p$ give vectors in θH^2 after applying the Toeplitz operator. The remaining term gives

$$[T_\varphi, H_\psi](z^{p-1} \theta) = -a_{-p} H_\psi(S^* \theta).$$

We claim $S^* \theta \notin \theta H^2$. If $\theta(0) \neq 0$ and $S^* \theta = \theta g$, then $\theta - \theta(0) = z \theta g \in \theta H^2$, so the nonzero constant $\theta(0)$ lies in θH^2 , which is impossible for a nonconstant inner function. If $\theta(0) = 0$, let $s \geq 1$ be the order of its zero at the origin. Then $S^* \theta = \theta/z$ has a zero of order $s - 1$, while every element of θH^2 vanishes to order at least s . Hence $S^* \theta \notin \theta H^2$ in either case. Therefore

$$H_\psi(S^* \theta) \neq 0, \quad [T_\varphi, H_\psi](z^{p-1} \theta) \neq 0.$$

For each $k < p$, the vector $z^{p-1} \theta$ belongs to $z^k \theta H^2$. The restriction cannot vanish there. This proves the equivalence. $\square$

A concrete example illustrating the sharp threshold established in Theorem 6 is provided in the Examples section (see Example 4).

Remark 2. Formula (5) admits a simple interpretation. If $m = d_-(\psi)$, then H_ψ annihilates $z^m H^2$. If $p = d_-(\varphi) \geq 1$, the coanalytic part of φ can lower the degree of a monomial by at most p . Hence the action of T_φ on every monomial in $z^{p+m} H^2$ produces only monomials belonging to $z^m H^2$, which are therefore annihilated by H_ψ . At the preceding level, $[T_\varphi, H_\psi]z^{p+m-1} = -a_{-p} b_{-m} \neq 0$. If φ is analytic and nonconstant, then $p = 0$, and the corresponding critical relation is $[T_\varphi, H_\psi]z^{m-1} = b_{-m}(\varphi - \varphi(0)) \neq 0$. Thus, in both cases, the threshold in (5) is sharp.

6. EXAMPLES

We conclude the main discussion with several examples illustrating the different cases of Theorem 4.

Example 1. Let

$$\varphi(z) = 2z^5 - z^2 + 3 - z^{-1} + 4z^{-3} \quad \text{and}$$

$$\psi(z) = z^4 + 2z + 1 - 3z^{-2} + 5z^{-4}.$$

Then

$$d_-(\varphi) = 3, \quad d_-(\psi) = 4.$$

Hence $d_{TH}(\varphi, \psi) = 7$. Equivalently,

$$[T_\varphi, H_\psi]|_{z^k H^2} = 0 \quad \Leftrightarrow \quad k \geq 7.$$

Note that at $k = 6$, the commutator does not vanish since $[T_\varphi, H_\psi]z^6 = -(4)(5) = -20 \neq 0$.

Example 2. Let

$$\varphi(z) = 1 + 2z - z^3 + 4z^5 \quad \text{and}$$

$$\psi(z) = 2 + z^2 - 3z^{-1} + 7z^{-4}.$$

Since φ is analytic and nonconstant and $d_-(\psi) = 4$, Theorem 3 gives $d_{TH}(\varphi, \psi) = 4$. Indeed, $[T_\varphi, H_\psi]z^3 = 7(\varphi - \varphi(0)) = 7(2z - z^3 + 4z^5) \neq 0$,

whereas the commutator vanishes on $z^4 H^2$.

Example 3. Consider

$$\varphi_1(z) = z^8 + 2z^2 + 1 + z^{-1} - 2z^{-3},$$

$$\psi_1(z) = 3z^5 - z + 2 + 4z^{-2} + z^{-4},$$

and

$$\varphi_2(z) = 100z^{10} - 7z + 5 + 11z^{-2} + 9z^{-3},$$

$$\psi_2(z) = z^7 + 6 - 13z^{-1} + 2z^{-3} - 8z^{-4}.$$

Although the coefficients and analytic degrees of the two pairs are different, their coanalytic degrees coincide: $d_-(\varphi_1) = d_-(\varphi_2) = 3$, and $d_-(\psi_1) = d_-(\psi_2) = 4$. Consequently, $d_{TH}(\varphi_1, \psi_1) = d_{TH}(\varphi_2, \psi_2) = 7$. This example confirms that the Toeplitz–Hankel commutativity depth depends only on the coanalytic degrees of the two symbols, as established in Corollary 4.

Example 4. (A nonmonomial degree-two Blaschke product)

Fix $0 < a < 1$ and set

$$\theta(z) = \frac{z(z-a)}{1-az}, \quad \psi(z) = \frac{1}{z(z-a)}, \quad \varphi(z) = z^{-2}.$$

Here $\psi \in L^\infty(T)$; its poles are strictly inside the unit circle, so it is not being asserted that ψ is analytic in the disk. Under the convention $H_\psi = UP_\psi M_\psi$,

$$H_\psi h = 0 \iff P_-(\psi h) = 0 \iff h(0) = h(a) = 0.$$

The last equivalence follows because the poles of $\frac{h}{z(z-a)}$ are removable exactly when h vanishes at both points. As $1-az$ and its reciprocal are bounded analytic multipliers,

$$\ker H_\psi = z(z-a)H^2 = \theta H^2.$$

For $h = z^2\theta g$, we have $H_\psi h = 0$ and $S^{*2}h = \theta g \in \ker H_\psi$. Hence

$$[T_{z^{-2}}, H_\psi]|_{z^2\theta H^2} = 0.$$

At the critical vector $h = z\theta$,

$$S^{*2}(z\theta) = \frac{\theta}{z} = \frac{z-a}{1-az},$$

$$\psi S^{*2}(z\theta) = \frac{1}{z(1-az)} = z^{-1} + \frac{a}{1-az}.$$

The negative Fourier part is z^{-1} , and the flip sends it to 1. Thus

$$[T_{z^{-2}}, H_\psi](z\theta) = -H_\psi S^{*2}(z\theta) = -1 \neq 0.$$

Since $z\theta \in z^k\theta H^2$ for $k = 0, 1$, we obtain

$$[T_{z^{-2}}, H_\psi]|_{z^k\theta H^2} = 0 \iff k \geq 2.$$

7. CONCLUDING REMARKS

We have determined the exact members of the monomial Beurling chain

$$\{z^k H^2 : k \in \mathbb{N}_0\}$$

on which the restricted Toeplitz–Hankel commutator vanishes for bounded symbols with polynomial coanalytic parts. When the Toeplitz symbol has a nonzero coanalytic part of degree p and the Hankel symbol has coanalytic degree m , the sharp

vanishing threshold is $p + m$. For a nonconstant analytic Toeplitz symbol, the corresponding threshold is m . Consequently, for nonconstant Toeplitz symbols and nonzero Hankel operators whose symbols have polynomial coanalytic parts,

$$d_{TH}(\varphi, \psi) = d_-(\varphi) + d_-(\psi).$$

We have also determined the rank of the restricted commutator along the monomial Beurling chain. Beyond the finite-degree setting, we have shown that the commutativity depth is infinite when the Toeplitz symbol is nonconstant and analytic and the Hankel symbol has infinitely many nonzero negative Fourier coefficients. Furthermore, under the hypothesis that the kernel of the Hankel operator is generated by a nonconstant finite Blaschke product, we have established a sharp vanishing threshold on the corresponding Beurling subspaces.

The characterization of finite commutativity depth for general nonanalytic Toeplitz symbols and Hankel symbols with infinitely many nonzero negative Fourier coefficients, as well as extensions to more general inner-function subspaces, remain questions for future investigation.

REFERENCES

- [1] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, vol. 1. Providence, RI: American Mathematical Society, 2002. doi: <https://doi.org/10.1090/surv/092>.
- [2] V. V. Peller, *Hankel Operators and Their Applications*. New York: Springer, 2003. doi: <https://doi.org/10.1007/978-0-387-21681-2>.
- [3] A. Brown and P. R. Halmos, “Algebraic properties of Toeplitz operators,” *J. Reine Angew. Math.*, vol. 213, pp. 89–102, 1964. doi: <https://doi.org/10.1515/crll.1964.213.89>.
- [4] S. Axler and Z. Čučković, “Commuting Toeplitz operators with harmonic symbols,” *Integral Equations Oper. Theory*, vol. 14, pp. 1–12, 1991. doi: <https://doi.org/10.1007/BF01194925>.
- [5] C. M. Liu, “Commutativity of Toeplitz operators on the Bergman spaces of the unit disk,” *Oper. Matrices*, vol. 14, pp. 857–870, 2020. doi: <https://doi.org/10.7153/oam-2020-14-53>.
- [6] Y. Shi, Y. Li, B. Zhang, and Y. Lu, “Commuting Toeplitz operators on the 2-analytic Bergman space,” *Acta Math. Sin. (Engl. Ser.)*, vol. 41, pp. 1855–1867, 2025. doi: <https://doi.org/10.1007/s10114-025-3195-5>.
- [7] X. Ding and D. Zheng, “Finite rank commutator of Toeplitz operators or Hankel operators,” *Houston J. Math.*, vol. 34, no. 4, pp. 1099–1119, 2008.
- [8] O. M. Alsalhi, “Commutativity of Hankel operators on the Hardy space,” *Mathematics*, vol. 14, 3103, 2026. doi: <https://doi.org/10.3390/math14173103>.
- [9] R. A. Martínez-Avendaño, “When do Toeplitz and Hankel operators commute?,” *Integral Equations Oper. Theory*, vol. 37, pp. 341–349, 2000. doi: <https://doi.org/10.1007/BF01194483>.
- [10] K. Guo and D. Zheng, “Essentially commuting Hankel and Toeplitz operators,” *J. Funct. Anal.*, vol. 201, pp. 121–147, 2003. doi: [https://doi.org/10.1016/S0022-1236\(03\)00100-9](https://doi.org/10.1016/S0022-1236(03)00100-9).
- [11] X. Ding, “The finite rank perturbations of the product of Hankel and Toeplitz operators,” *J. Math. Anal. Appl.*, vol. 337, pp. 726–738, 2008. doi: <https://doi.org/10.1016/j.jmaa.2007.01.113>.
- [12] R. E. Curto, G. Datt, and B. B. Gupta, “Commutativity of Hankel and Toeplitz operators on the Hardy space of the n -torus,” *Bull. Sci. Math.*, vol. 194, 103466, 2024. doi: <https://doi.org/10.1016/j.bulsci.2024.103466>.
- [13] H. Zou and T. Yu, “Commuting Hankel and Toeplitz operators on the Hardy space of the bidisk,” *Adv. Oper. Theory*, vol. 11, 41, 2026. doi: <https://doi.org/10.1007/s43036-026-00513-w>.

- [14] A. Beurling, "On two problems concerning linear transformations in Hilbert space," *Acta Math.*, vol. 81, pp. 239–255, 1949. doi: <https://doi.org/10.1007/BF02395019>.
- [15] P. Aroda, A. Chattopadhyay, and S. Jana, "Restricted Toeplitz and Hankel operators," arXiv:2603.17409, 2026. doi: <https://doi.org/10.48550/arXiv.2603.17409>.